The Theory of Embedding Fields with Applications to Psychology and Neurophysiology (I)

by

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PREFACE

The theoretical developments in this monograph were originally motivated by a study of standard experimental effects in human learning situations. These effects have received extensive experimental consideration by many authors over at least a forty year period. Once the original theoretical structure was completed, it imposed natural theoretical extensions that generated familiar experimental effects from a number of other fields of experimental inquiry in psychology, neurophysiology, and neuroanatomy. Each of these further fields is receiving and has received intensive experimental consideration as well.

The enormous literature of the experimental contributions relevant to the theoretical structure of the paper has made the author's task of accurately expressing appreciation for the stimulation of others' contributions a formidable one whose completion, were it possible, would have itself substantially increased the length of the monograph. The task is further complicated by the fact that the work has proceeded by constructing a dynamical system directly from a concentrated body of stable data and then passing in a purely theoretical way to a much larger frame to independently derive results which later were either found to have been conjectured or found experimentally by others, or are new.

Since the purpose of the monograph is to present a unifying conceptual frame which is not tied to the work of any single group of contributors, experimental effects are presented as manifestations of deeper dynamical principles. Since each of these effects, when known, has often been studied by many authors, I have found it necessary to assume that it is well known and mention a particular author's name only when he is inextricably tied to the effect. Extensive bibliographies of references are available in the broad literature of periodical reviews of experimental progress, and will hopefully offset the necessary bibliographical omissions.

My debt to others, whether direct or indirect, is great nonetheless, for a theoretical monograph of this kind would have been quite inconceivable were it not preceded by a long period of vigorous and intelligent experimentation.

Section Titles

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with Applications to Psychology and Neurophysiology (1)

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1. Introduction

Much of the mathematical formalism of contemporary psychological theory, for example stimulus sampling theory, consists of a collection of probability formulae which are derived from arguments concerning certain simple operations on abstract point sets. Such a theoretical structure allows one to express several intuitive conceptions of psychological processes in a manner that maintains a close rapport, in principle, with those aspects of collected data that fall within the relatively narrow confines of the theoretical postulates. These uneven virtues have brought gratifying initial success in predicting a few interesting characteristics of data in several simple experimental situations.

At least three related limitations must, however, be observed in the foundations of the theory that transcend any particular predictive criterion. First, the sets over which the theory's formulas lie are merely formal and without any apparent relation to psychologically important structures in the real world. Second, the probability statements that are found in the theory may be formulated in such a way that all of their variables take values exclusively in data sheets giving counts of evoked stimuli and responses. The formal structure of the theory is hereby inextricably tied to simple daily evaluations of the external guises of learning processes and provides little insight into the structure of these processes. Although this simplicity of attack makes possible the proliferation of models to fit a limited number of classes of learning data, it would appear to have also left the impression that when a given model does not serve adequately in a fixed empirical setting, the most profitable goal for the theorist should be to find another model whose foundations stem quite directly from those of those of the first and whose slightly altered equations and thoughtfully chosen parameters render a closer fit to data more feasible. This impression has persisted in spite of convincing evidence that very slight generalizations of theoretical postulates generate highly nonintuitive formulas of considerable complexity, and that the theory provides no principles by which to extend the simplest formulas to really interesting cases - a third and imposing difficulty.

The present paper entirely abandons this rationale of theory construction and searches for new theoretical foundations in the belief that the extant theories really do not grasp the underlying significance of complicated psychological data in a simple and coherent way. The outcome of this search is a new mathematical structure which overcomes many of these difficulties. Fundamental to the motivation of the new theory is the realization that the dynamics of many psychological problems may be viewed from a unified point of view once the geometrical substructures that characterize each separate problem are elaborated and distinguished. The interrelations between these divers geometrical substructures and their similar dynamical superstructures on the psychological level can be gradually extended and refined until the sense in which they are averages over the fine geometrical and dynamical structure of living brains becomes clear. This passage from psychological to neurological structures is the passage from the macroscopic variables to the microscopic variables needed to understand neural interactions. It is natural to begin our study with macroscopic variables since behavioral interactions proceed on a macroscopic level. Correspondingly, many neurological properties of higher brains are of a beautifully sculpted global character which cannot easily be viewed in their functionally unitary form by contemporary microscopic neurological techniques. The process of passing from macroscopic to microscopic variables shall establish a connection between the psychological and neurological approaches which shall help to illuminate the results of each approach and, hopefully, to unify the course of future studies.

The present paper shall begin by studying in a leisurely and qualitative manner a particular type of psychological data, namely from verbal learning.

Little effort shall be made to give the most concise presentation, since the theoretical ideas are new and one cannot fully understand the value of a new theoretical tool before seeing how smoothly it applies to many interesting examples. On the other hand, since the data in verbal learning is so bountiful and is usually well-replicated, it is impractical to give bibliographies after every theoretical analysis. Rather, it shall be tacitly assumed that every example which we consider represents a central empirical finding unless otherwise stated.

In order to initiate our theoretical study, it shall be necessary to perform an inductive leap to a mathematical system which is, in a natural sense, the simplest system that provides nontrivial insight into the data at large. Such a leap is never particularly easy to motivate at first since it originally arises as a direct perception of the underlying structure of the data. It shall therefore be initially justified by showing that the mathematical object which arises actually gives a qualitative description of a large quantity of data from a small number of principles. After this has been accomplished, some simple intuitive questions and an obvious formal difficulty in the original mathematical system shall impose a natural extension of its structure. Once this extension is before us, a very broad collection of verbal learning data, including several language phenomena of higher type, shall immediately be seen to be within the grasp of the theory. But even more shall be gained in this step, for through it the significance of the dynamical system under study as a representation of global neurological events shall slowly emerge. Indeed, the paper shall proceed from this point to successively more refined representations of neurodynamics and neuroanatomy by extending the original mathematical system in a rational way. The outcome is an explanation of many phenomena in general learning theory, psychophysics, perception, and the more properly neurological sciences. Along the way, some light will be thrown on such topics as the uses and limitations of information theory in psychology, spectral analytic properties of neurological systems, the specific inadequacies of the computer analogy of neural processing, and the weaknesses of stimulus sampling theory which prevented its generalization to a broad variety of experimental situations. The controversies between Contiguity and Gestalt learning theorists, and between "all-or-none" and "continuous" learning theorists will also be qualitatively resolved, and the work of such people as Hull and Guthrie will be understood from a higher vantage point. Among the neurological facts to be discussed are the EEG rhythms and their significance, thalamo-cortical interactions, cortico-cerebellar interactions, cortico-cortical interactions, lateral inhibitory processes, pre- and post-synaptic habituation, spatial and temporal masking, chemical transmitters and coenzymes, the coupling of membrane polarity to cellular control macromolecules and their repression, neural Use and Disuse, axonal frequency modulation, neural "power laws", the dynamics of neuron pools, plasticity in the growth of neurons, various uses of neural thresholds, temporal and spatial summation effects, cortical conditioning, the cellular distributions in neocortex, cerebellum, and thalamic nuclei and their significance, the interaction of specific and nonspecific regions such as the reticular formation, glial-neuron interactions, the behavior of sequences of specific sensorimotor relay nuclei, and related anatomical and physiological matters. The conclusion of this development is the statement of a general theoretical program carried by a small collection of equations and principles which have a clear neurological interpretation, and which bring together many branches of the psychological and neurological sciences into a harmonious perspective.

2. A Free Embedding Field for Homogeneous Serial Verbal Learning

Thus let us begin with a special empirical situation whose more subtle complexities shall initially be ignored to achieve a tractable though idealized theoretical structure. In particular, suppose that we are given a classical serial rote learning situation with nonsense syllables as the verbal units. For simplicity it is desirable to assume that all units have equivalent theoretical status, so suppose that each syllable consists of a single consonant and that the consonants are matched along a variety of scales, say in associative value, familiarity, and the like. Further assume that the list is n consonants long, and that learning is by the serial anticipation method with visual presentation, followed by an immediate utterance of the stimulus item, and verbal recall. One may imagine that the subject sits before a window in which are sequentially presented the consonants of the list according to a predetermined cyclic plan. It is a matter of daily experience that such verbal units are intuitively perceived as unitary or simple; that is, one never seeks in actual discourse to decompose a given consonant into smaller verbal parts. Thus it is intuitively natural to let each consonant be a fundamental and indivisible mathematical entity. The interpretation of such a representation shall soon be clearer but at present we merely associate in good faith to each consonant r_i an abstract point p_i, where the subscript indicates the order of presentation in the list. Since the empirical data of this problem in verbal learning is a series of transitions through time from one point (consonant) to another, suppose that the points p_i are the nodes of a completely connected directed graph; that is, from p_i to p_j draw the directed line l_{ij}, and continue this process for all ordered pairs of points (see diagram 1). This simple structure adequately serves to emphasize the principal first-order structural features of the process. It shall be called the <u>structural carrier (of the embedding field)</u> associated with this problem.

The structural carrier represents the geometry of the problem, but it does not in any way encompass the time dependent behavior so familiar in the use of language. In order to discuss this dynamical behavior, it is necessary to define a variety of time-dependent functions that are naturally associated with the points and lines of the structural carrier. Thus, with each p_i associate a positive constant M_i, which is called the <u>total embedding space</u> of p_i. Further associate an (as yet undetermined) function s_i, $0 \leq s_i \leq M_i$, the <u>strength function of</u> p_i. To every line l_{ij}, assign two positive real numbers A_{ij} and p_{ij}, where A_{ij} is the <u>total embedding space of</u> l_{ij} and p_{ij} is the <u>structural connection number of</u> l_{ij}. Similarly denote the as yet undetermined function c_{ij}, $0 \leq c_{ij} \leq A_{ij}$, as the <u>strength function of</u> l_{ij}, and the function $\tilde{c}_{ij} \geq 0, \leq 1$ as the <u>renormalized strength function of</u> l_{ij}.

It is in terms of these functions that the dynamics of this situation shall first be described. The strength s_i of p_i shall be a quantity fluctuating in time over p_i. At every time t a certain function of $s_i(t)$ will be transmitted through the line l_{ij} and shall be received at the point p_j after a fixed transmission time lag t_{ij}. The effect of this transmitted quantity from p_i upon the activity of p_j shall in turn depend upon the value of s_j at the time at which the quantity arrives at p_j. Moreover, while this process of strength transmission through the l_{ij} proceeds, it shall influence the growth of the various functions c_{ij}. The values of these functions c_{ij} at future times shall in turn modulate future strength transmissions between the various points, via the $\tilde{c}_{ij}$ functions.

More explicitly, suppose that M_j represents a structural quantity that can be exhausted and that $s_j(t)$ represents the extent to which it is exhausted at time t. The value of s_j at a given time may also be viewed as a kind of activity that pervades a structure whose total extent is M_j. In particular, when $s_j = M_j$, p_j is maximally active. Now assume that if the line l_{jk} were operating at its maximal capacity, a quantity rs_jp_{jk}, $0 < r = \text{constant} < 1$ would be transmitted through it at any time. Thus p_{jk} represents a measure of the transmission capacity of l_{jk} and r is a factor which translates point strength values into transmitted strength values. Assume further that when the transmitted strength reaches the endpoint of the line l_{jk} it crosses a structure which alters its value by the multiplicative factor $\tilde{c}_{jk}$. If the time taken to transmit strength values from p_j to p_k over l_{jk} is t_{jk}, then the strength transmitted from p_j to p_k and received by p_k at time t is $r(s_j(t - t_{jk}))\, p_{jk}\tilde{c}_{jk}$, where the notation $s_j(t - t_{jk})$ merely denotes the evaluation of s_j at time $t - t_{jk}$. In particular, assume that $t_{jj} << \min_{k=j} t_{jk}$, so that as a rough estimate the approximation $t_{jj} = 0$ is made in this case. This is a special assumption that does not hold in more general situations. Let us now further assume that the strength values received from the entire set of points at a fixed p_k combine additively. Then the total transmitted strength received at time t at p_k is $T_k = r\,\Sigma_{i=1}^{n}\; s_i(t - t_{ik})\, p_{ik}\tilde{c}_{ik}$.

We already observed that the effect of strength transmitted to p_k in not independent of the activity at p_k at its time of arrival. In fact, we suppose that contributions of transmitted strength values will cause a growth in the time derivative of s_k that is proportional to the transmitted strength and the function $M_k - s_k$, which represents the extent to which the quantity M_k has not been exhausted at the time new transmitted strength arrives at p_k. It is for this reason that $E_k = \alpha(M_k - s_k)$, $0 < \alpha$, is called the <u>effective embedding space of</u> p_k. The choice $\alpha(M_k - s_k)$ for E_k is merely the simplest choice of an element in a more general class of effective embedding spaces which we shall encounter. In addition to the growth process hereby determined for s_k, a process of strength decay given by a function D_k is also envisaged to proceed locally at each p_k. The particular process here chosen is a simple exponential one. In mathematical form, the above rules may be written as:

$$\frac{ds_k}{dt} = E_k T_k - D_k, \qquad (1)$$

where

$$E_k = \alpha(M_k - s_k),$$

$$T_k = r \, \Sigma_{i=1}^{n} (s_i(t - t_{ik}))p_{ik}\tilde{c}_{ik},$$

and

$$D_k = \beta s_k, \qquad \text{for } k = 1, 2, \ldots, n,$$

where $\alpha, \beta > 0$, $0 < r < 1$, and we impose the condition (*):

$$\max_i \frac{\alpha r M_i p_{ii}}{\beta} < 1 \qquad (*).$$

That is, $\frac{ds_k}{dt} = \alpha^+ (M_k - s_k)(\Sigma_{i=1}^{n} s_i(t - t_{ik}) p_{ik}\tilde{c}_{ik}) - \alpha^- s_k$, where $\alpha^+ = \alpha r$ and $\alpha^- = \beta$ are the growth and decay constants of the equation.

To determine the behavior of the functions c_{ij}, suppose that c_{ij} represents a process that takes place at the end of the line l_{ij}. Call this terminal region the <u>node</u> N_{ij} <u>of</u> l_{ij}. Suppose that the dynamical behavior of N_{ij} is locally determined by the strength quantities which impinge on it from both p_i and p_j. In the present case, the only difference envisaged between p_i and p_j in determining this behavior is that the quantities representing the activity of p_i take longer to reach the vicinity of N_{ij} than those representing the activity of p_j. With this simple distinction between p_i and p_j in mind, the behavior of c_{ij}, too , may be viewed as the net effect of growth and decay processes over a localized geometrical structure. Indeed A_{ij} is, akin to M_k, an exhaustible structural quantity belonging to the node N_{ij}, while c_{ij}, akin to s_k, represents the total extent of an activation process that modulates the strength transmission from p_i to p_j via N_{ij} through time. A_{ij} is simply the maximum value which this process can attain. p_i transmits $r\, s_i(t - t_{ij})\, p_{ij}$ to N_{ij}, and N_{ij} immediately transfers $r\, s_i(t - t_{ij})\cdot p_{ij}\tilde{c}_{ij}$ to p_j, where $\tilde{c}_{ij}$ is a function of $\{c_{ik}\}$.

To determine c_{ij}, suppose at any time t that the time derivative of c_{ij} is proportional to the products of the point strength values transmitted from p_i and p_j and reaching N_{ij} at time t. By strength "transmitted" from p_j to N_{ij} we mean the contiguous relation of N_{ij} to p_j, which permits the assumption that s_j acts directly upon N_{ij}, up to a multiplicative factor. Notice that since the strength transmitted from p_i to N_{ij} is computed before it crosses N_{ij}, the factor $\tilde{c}_{ij}$ does not appear. Indeed, once N_{ij} is crossed, the transmitted value loses its identity as a product of the strength activity at p_i and becomes expressible totally in terms of s_j. The growth of the line connection function c_{ij} is hereby determined by the simultaneous activity of p_i and p_j, where simultaneity is measured relative to the time of arrival of transmitted values at N_{ij}, and the activity of the relevant points is reflected entirely by the strength values that they transmit to N_{ij}. Moreover, again in analogy with s_k, the growth rate of the line connection c_{ij} is influenced by its present state. In fact, define the quantity $E_{ij} \equiv u(A_{ij} - c_{ij})$, $0 < u$, in close analogy with E_k, and call E_{ij} the <u>effective embedding space of</u> N_{ij} (or, <u>of</u> l_{ij}). The growth process for N_{ij} is completely determined by requiring that the rate of growth of c_{ij} at time t be proportional to the product of the transmitted strength values received from p_i and p_j at time t, multiplied by the effective embedding space of N_{ij} at time t. To determine the decay process for lines, suppose that c_{ij}, which measures the activated portion of A_{ij}, decays at a rate D_{ij} proportional to the product of its value at time t with the values of the effective embedding spaces (the inactive regions) of p_i and p_j that impinge on N_{ij} at time t. Notice that the effective embedding space of p_i that impinges upon N_{ij} at time t is not proportional to $M_i - s_i$, but is rather proportional to $M_i - s_i(t - t_{ij})$; that is, one can think of a parallel transplantation of the inactive region of p_i at time s along the line l_{ij} until it reaches N_{ij} at time $s+t_{ij}$. This decay law can be heuristically summarized by saying that the inactive region of the effectively connected line structure of l_{ij} decays exponentially into its disconnected state. Or, still heuristically, lines which are not activated fall gradually into desuetude. These laws for the line structure have the following mathematical formulation:

$$\frac{dc_{ij}}{dt} = E_{ij}\,\hat{T}_{ij}\,\hat{\hat{T}}_{ij} - D_{ij}\,\hat{E}_{ij}\,\hat{\hat{E}}_{ij} \tag{2}$$

where

$$E_{ij} = u(A_{ij} - c_{ij}),$$

$$\hat{T}_{ij} = r\,s_i(t - t_{ij})\,p_{ij}, \quad \text{so that } T_{ij} = \hat{T}_{ij}\,\tilde{c}_{ij},$$

$$\hat{\hat{T}}_{ij} = \hat{r}\,s_j,$$

$$D_{ij} = \epsilon c_{ij},$$

$$\hat{E}_{ij} = \rho(M_i - s_i(t - t_{ij}))\,p_{ij},$$

$$\hat{\hat{E}}_{ij} = \hat{\rho}(M_j - s_j),$$

and u, r, r, $\epsilon, \rho, \hat{\rho} > 0$. That is,

$$\frac{dc_{ij}}{dt} = \gamma^{+}_{ij}(A_{ij}-c_{ij})s_i(t-t_{ij})s_j-\gamma^{-}_{ij}c_{ij}(M_i-s_i(t-t_{ij}))(M_j-s_j)$$

where $\gamma^{+}_{ij} = ur\hat{r}p_{ij}$ and $\gamma^{-}_{ij} = \epsilon\rho\hat{\rho}p_{ij}$ are the growth and decay constants of the equation. We also assume that the growth equation for the line functions is more slowly varying than that for the strength functions.

3. Experimental Inputs.

The collection of equations (1) and (2) together constitute the free embedding field equations for this problem (diagram 2). A free embedding field is thus a pair consisting of a structural carrier and a set of functions of the above kind. Such a field is unperturbed by external influences and, as one sees from condition (*), increasingly small strength quantities are transmitted from point to point while the c_{ij} functions gradually decay. In fact, $\lim_{t\to\infty} s_i(t) = \lim_{t\to\infty} c_{jk}(t) = 0$ for all i, j, and k. The free case is hardly ever realized during an active learning session, but it illustrates a pervasive tendency in embedding fields to achieve an undifferentiated and quiescent state. In realistic cases, two general categories of external disturbances influence the field behavior. The first of these is the category of experimental inputs $I^{(e)}_i = I^{(e)}_i(t)$, $i = 1, 2, \ldots n$, which correspond, for example to the presentation to an attentive learning subject of a stimulus item r_i that has a point representation p_i in the embedding field. The second category is that of the internal inputs $I^{(in)}_i = I^{(in)}_i(t, s_1, \ldots, s_n, c_{11}, c_{12}, \ldots, c_{km}, \ldots)$, $i = 1, 2, \ldots n$, which correspond to dynamical interactions between the given embedding field and other field structures which together represent the total on-going psychological behavior of the individual. The internal inputs are sometimes called feedback inputs for this reason. We assume that these two kinds of input influence their respective points in a manner formally identical to that of the transmitted strength factors; namely,

$$\frac{ds_i}{dt} = E_i (T_i + I_i^{(e)} + I_i^{(in)}) - D_i .$$

In this sense, the quantities T_i, $I_i^{(e)}$, and $I_i^{(in)}$ all behave as external influences on local point activity. In order to geometrically represent these additional features of the field, the structural carrier is enlarged by the addition to each point of a pair of directed lines which terminate at that point, but whose initial points are as yet undetermined (diagram 3).

The functions $I_i^{(in)}$ shall temporarily be ignored. They are quite complicated. The functions $I_i^{(e)}$ are, however, amenable to an elementary treatment, which will be gradually extended as we proceed. Thus suppose that an experimental determination of the times at which verbal units will be presented to a subject is made. Let s_{ij} be the time at which the j^{th} presentation of the i^{th} item is made. Further let

$$J_i^{(e)}(t) = \begin{cases} 0, & t < 0 \\ a_i^{(e)} t \cdot \exp(-b_i^{(e)} t), & 0 < t \leq T^{(e)} \\ 0, & T^{(e)} < t \end{cases}$$

where $a_i^{(e)}, b_i^{(e)} > 0$ are fixed constants, and require that

$$I_i^{(e)}(t) = \Sigma_j J_i^{(e)}(t - s_{ij}).$$

Each $J_i^{(e)}(t - s_{ij})$ is called an <u>experimental input with onset time</u> s_{ij}. Every experimental presentation of verbal unit r_i to which the subject attends is conceptualized as an input to p_i of a fixed functional form whose onset time is the time of presentation, and successive inputs to p_i

combine additively. Actually, the onset time of experimental inputs to the embedding field lags behind the experimental stimulus onset by a very brief time interval. We assume this interval to be the same for all points and to be invariant under successive experimental manipulations, in keeping with the simplifying law that individual experimental inputs combine additively. The entire onset time sequence is thus translated by an inessential small constant. In fact, the functions representing successive experimental inputs to a fixed point are not strictly equal to the translates in time of a single function $J_i^{(e)}$; nor do the various experimental inputs strictly add. We can, however, safely ignore these matters in our present elementary discussion since the distribution of onset times is highly regular in the classical experimental paradigms and is large relative to a simple reaction time.

4. $\tilde{c}_{ki} = ?$

A determination of $\tilde{c}_{ki}$ is now possible. The simplest such choice is obviously $\tilde{c}_{ki} = c_{ki}$, and corresponds to the supposition that the effects of the transmission process through any fixed node are independent of the comparable process at any other node. Such a supposition is not entirely correct, however. In particular, for this choice of $\tilde{c}_{ik}$ no nontrivial list could ever be learned perfectly. Since a considerable amount of discussion must be given before the actual situation can be adequately revealed, we shall for the present make a somewhat artificial, but nonetheless highly illuminating, choice of $\tilde{c}_{ki}$ in a special case. Thus, let $M_i = 1$, and $A_{ii} = B$ for all i, while $A_{ij} = A$, $i \neq j$; that is, $A_{ij} = B(1 - \delta_{ij}) + A\delta_{ij}$, where δ_{ij} is the Kronecker delta. Further let $t_{ij} = v(1 - \delta_{ij})$, $0 < v$, and let $p_{ij} = \lambda(1 - \delta_{ij}) + \mu\delta_{ij}$, $0 < \mu, \lambda \leq 1$. For this situation, the choice $\tilde{c}_{ki} = c^*_{ki}$, where $c^*_{ki} = c_{ki}/\Sigma_r c_{kr}$, is in fact a closer approximation to actual events than c_{ki}. The significance of the choice c^*_{ki} is, for example when $\lambda = \mu = 1$, that the total strength transmitted from p_k is given by $\Gamma_k = r\Sigma_i\, s_k\, p_{ki}\, \tilde{c}_{ki} = rs_k$. The line structure determines a continually varying partitioning of the transmittable strength at a point to the other points in the embedding field whenever the c values leading from a given point are normalized. Let this choice of $\tilde{c}_{ki}$ be assumed unless the contrary is stated in the following qualitative discussion of several of the psychological underpinnings of the embedding field equations.

5. An Example of Directionality of Associations

Suppose for simplicity that the embedding field contains only four points: $p_1, \ldots, p_4$, and that the above special uniformity assumptions are in force; that is, $M_i = 1$, $A_{ij} = B(1-\delta_{ij}) + A\delta_{ij}$, and so on. Assume further that $\mu = 0$ and $\lambda = 1$, or $p_{ij} = 1-\delta_{ij}$. Choose all initial values for the c_{ij}, $i \neq j$, to be equal and small, and suppose that $c_{ii}(0) = 0$ for all i. Also let all $s_i(0) = 0$. We are hereby given a highly homogeneous and quiescent four point field, free from self-excitations. Now deliver an external input to point p_1. p_1 then transmits equal measures of strength to p_2, p_3, and p_4, whence the rate of growth of each of the c_{1j}, j = 2, 3, 4, will be the same, by symmetry, and the path strengths c^*_{1j}, j = 2, 3, 4, will remain identically constant. Moreover, by $rp_{1j} < 1$, s_1 will be larger than $s_2 = s_3 = s_4$ during a time interval following input onset to p_1 after which all s_i are small. Deliver an input to p_2 during this interval. Following this delivery, the path strength c^*_{12} will grow to the equal detriment of c^*_{13} and c^*_{14}, since $s_3 = s_4 < s_2$. Similarly, the path strength c^*_{21} will grow significantly more than c^*_{23} and c^*_{24} if the s_2 values, now augmented by an input, are transmitted to the p_i, i = 1, 3, 4, before s_1 decays to the $s_3 = s_4$ range of values. This distribution of values in the c^*_{2i} depends critically on the relative sizes of the transmission times $t_{ij} = v$, the interval Δ during which s_2 is large, and the interval Λ_{12} between the onset of the input to p_1 and the input onset to p_2. For example, if $\Delta << v$, then we can choose Λ_{12} in such a way that strength transmitted from p_2 to p_1 after the input onset at p_2 arrives at p_1 when s_1 is very small. Since the strength originally transmitted from p_1 to p_3 and p_4 remained within l_{13} and l_{14} for v time units, Λ_{12} can also be adjusted so that the p_2 transmissions to p_3 and p_4 arrive before their strength functions have decayed too much. Consequently, c_{21}^* will decay to the equal advantage of c_{24}^* and c_{23}^*, which is the reverse inequality of that obtained above. In fact, however, $\Delta_{12} << v$ never occurs in these fields. If it did, the probability that transmitted excitation from some point arrives at another point while the latter is being excited by an external input would be very small, whence the field would remain practically homogeneous, and uninteresting, for all time. The inequality $c_{21}^* > c_{23}^* = c_{24}^*$ may therefore be safely assumed. A more precise understanding of the relative temporal magnitudes underlying it will come in the following pages.

Now let Q_{ij} be the time between the input onset at p_j and the first time that strength induced by the input to p_i reaches p_j. $Q_{12} = |\Lambda_{12} - v|$ while $Q_{21} = \Lambda_{12} + v$. Since $Q_{12} < Q_{21}$, it is usually true that $c_{21}^* < c_{12}^*$. Once $c_{21}^*(t_0) < c_{12}^*(t_0)$, the inequality will tend to preserve itself for times $t > t_0$, since transmittable strength from p_1 will be partitioned in greater quantities to p_2 than will transmittable strength from p_2 be partitioned towards p_1.

We therefore see that if a pair of inputs is delivered successively to p_1, then to p_2, in a homogeneous and quiescent four point field whose constants are adjusted to avoid trivialities, c_{12}^* and c_{21}^* will grow to the detriment of the line values of the other points in the field, with the advantage going to c_{12}^* (diagram 4). Suppose that two four point fields $\mathcal{F}_1$ and $\mathcal{F}_2$ are now given which are identical replicas of one another and have received the successive inputs to p_1 and p_2. Let t_o be the onset time of the input to p_2 in both fields. If at time $t_1 > t_o$, r_1 is presented to $\mathcal{F}_1$ again and c_{12}^* is still relatively large, an external input to p_1 will be delivered and the strength transmitted from p_1 will pass in greatest quantity to p_2, whence s_2 will grow larger than s_3 or s_4. Similarly, if at t_1 an input is delivered to the p_2 of $\mathcal{F}_2$ rather than to its p_1, s_1 will grow larger than s_3 or s_4 due to the favorable normalization of c_{21}^*. Nonetheless, the inequality $c_{12}^* > c_{21}^*$ implies that the maximum value of s_2 achieved in $\mathcal{F}_1$ due to transmission induced by the second input to p_1 will exceed the maximum of s_1 in $\mathcal{F}_2$ due to transmission induced by the second input to p_2. We now suppose that increasing the value of s_k at any time enhances the probability that the subject will offer r_k as a response at that time. The previous argument then shows that in a homogeneous four point field, the presentation of r_1, then r_2, followed after a pause by r_1 will lead to the evokation of r_2 with high probability. Similarly, r_1, then r_2, followed after a pause by r_2, will yield r_1 with fairly high probability. But whenever the two pauses are of equal length, the probability of evoking r_2 in response to r_1 might well exceed the probability of evoking r_1 in response to r_2.

We now assume that the c_{ij}^* function plays the role usually ascribed to the ill-defined concept of the "strength of association" from r_i to r_j through time. We also subscribe to the following standard terminology: the temporally "forward" direction in a list is the direction in which new inputs arrive. In particular, p_{i+1} is more forward than p_i in this sense. The temporally "backward" direction is the reverse of the forward direction. Using this familiar terminology, it may be said that contiguous inputs to two points in a symmetrical four point field cause associations to be formed in both the temporally backward and temporally forward direction. The forward direction is however, usually preferred.

6. An Example of Symmetry of Associations.

Notice that although we have imagined a four point field, external inputs were only delivered to the two points p_1 and p_2. What happens if we strip away the points p_3 and p_4 to which inputs were never delivered? Had we originally been given a two point field with points $\{p_1, p_2\}$ and otherwise identical initial conditions, $c_{12}^* = c_{21}^* \equiv 1$ for all time. We can interpret this fact to mean that the evokation of r_2 follows the presentation of r_1 with certainty, and vice versa, which shows that the two point field maximizes the symmetry of forward and backward structuring possibilities among all possible embedding fields. To an observer who can detect only the presentation of external inputs and the evokation of overt responses, this comparison of two and four point fields must be a cause of considerable chagrin. For it shows that the presentation of an identical array of external inputs to two different embedding fields can yield strikingly different results in the overt response paradigm. Since our imaginary observer cannot see the interpolated fields, he must woefully assume that equal stimulus conditions do not generate equal response distributions. Such an observer is in a hopeless situation, for he does not yet have a complete set of variables at his disposal with which to discriminate the two situations. He must come to realize that the manner in which a set of points $\mathcal{P}$ is embedded in a larger point field $\hat{\mathcal{P}}$ can strongly alter the dynamical behavior associated with $\mathcal{P}$ even when the additional points $\hat{\mathcal{P}} \setminus \mathcal{P}$ are excited only by indirect transmitted excitations from $\mathcal{P}$ and are entirely unperturbed by external inputs.

7. The Immersion of Subfields, the Stability of Local Dynamics, and Effective Dynamical Contractions.

In the laboratory, one is usually confronted with subjects who already possess a working command of an entire language, not merely of two or four points. The comparison between two and four point fields shows that we cannot hope to ultimately ignore the many thousands of points representing the language units in the subject's command without seriously altering our theoretical conclusions. The desire to conceptually decompose a given field into arbitrary subfields of lesser complexity, to study these subfields thoroughly, and to return to the total field by simply pasting the pieces together is seen to be impossible from the outset. We have no linearized approximation at our disposal. We must, rather, always expect in the following pages that as increasingly refined considerations come before us, our old conclusions will show themselves as merely first approximations. The alternative is to present a very complicated nonlinear system at the beginning, without giving the slightest hint of how one can naturally come to understand it.

To avoid the question of how point sets $\mathcal{P}$ come to be embedded in larger sets $\hat{\mathcal{P}}$ as a subject's learning experience grows, we suppose that the field which we are considering at any moment represents an idealization of the entire point field that interacts with a given class of external inputs. The spontaneous growth of new points is a hard question to which we shall return in gradual steps. The comparison of two and four point fields raises another question to which we can turn immediately with profit. This question can be phrased in several equivalent ways: Under what external input conditions will the dynamics of a large point field approximate the highly stable and symmetric dynamics of a two point field? Under what conditions will the embedding of a given point set into a larger set cause little change in the local dynamics of the original set? If new points are constantly being added to a subject's total point field as his experience grows through time, how does it happen that the subfields representing the learning of old material are not entirely disrupted by the addition of new points and their interactions?

Imagine an embedding field with $n >> 2$ points and practically the same homogeneous initial conditions as before. Alter the condition on p_{ij} by requiring that

$$p_{ij} = \begin{cases} 1 & j = i+1 \pmod{n} \\ 0 & \text{otherwise} \end{cases}$$

where the symbol $k \pmod{n}$ denotes the counting of integers modulo n. This condition may also be written $p_{ij} = \delta_{j,\, i+1 \pmod{n}}$ (diagram 5). (A comparable analysis to the following can be achieved when $j = i+1 \pmod{n}$ is replaced by $j \in U_i$ = any small ($<<n$) subset of indices.) In this situation, $p_{i \pmod{n}}$ transmits strength only to $p_{i+1 \pmod{n}}$. Deliver an input to a fixed point, say p_1. Observe that the maximum strength transmitted from p_1 to p_2 at any time is less than or equal to $r < 1$. The strength transmitted to p_3 via p_2 as a result of the input to p_1 is of the order r^2 (<1) at most. Similarly, the transmitted strength to p_k derived from the input to p_1 over the transmission chain $p_1 \rightarrow p_2 \rightarrow \cdots \rightarrow p_k$ is of the order r^{k-1} at most, whenever $k < n$. Contributions that arise from higher order transmissions of the form $p_1 \rightarrow p_2 \rightarrow \cdots \rightarrow p_n \rightarrow p_1 \rightarrow p_2 \rightarrow \cdots \rightarrow p_k$ may be ignored since n is large. It is general rule that higher order transmissions through a cyclic structure are killed off with successive transmission steps at a rate that is at least multiplicative.

Notice that a comparable sequence of transmissions could have been achieved by setting $p_{ij} = 1 - \delta_{ij}$ and $c^*_{i,\,i+1(\mathrm{mod}\ n)}(0) \approx 1$, for strength transmission is determined by the product $p_{ij}c^*_{ij}$. In either case, since the strength transmitted to p_n after an input to p_1 is dominated by $r^{n-1} << 1$, an observer at p_n could easily be deceived into believing that no external input had been delivered to the field, even though p_n is strongly connected to p_1 by the line l_{n1}. The structural connections p_{ij} and the line strength functions c^*_{ij} exercise an extraordinary control in their interaction with fixed inputs on the distribution of strength throughout the point field.

Suppose that we are given a field in which these line functions are initially so determined that all inputs to the field arrive at a point set $\mathcal{P}$ which transmits practically no strength to the point set Q, where $\mathcal{P} \cap Q = \emptyset$, and the point strength values of Q are very low at the outset. In the previous example, we can choose $\mathcal{P} = \{p_1, p_2, \ldots, p_{n-1}\}$ and $Q = \{p_n\}$, and can let the collection of external inputs be the input to p_1. Then Q may safely be ignored when studying the dynamics induced by the external inputs to $\mathcal{P}$. Or what is the same, $\mathcal{P}$ can be embedded in $\mathcal{P} \cup Q$ without seriously altering the local dynamics of $\mathcal{P}$. The line functions p and c^* have determined an <u>effective dynamical contraction</u> of $\mathcal{P} \cup Q$ to $\mathcal{P}$ relative to the given external input paradigm to $\mathcal{P}$. Notice that the choice of input paradigm is critical in determining an effective dynamical contraction. In the case $\mathcal{P} = \{p_1, p_2, \ldots, p_{n-1}\}$ and $Q = \{p_n\}$, for example, if inputs are delivered to p_{n-1}, then p_n may no longer be excluded from the set of points which nontrivially influence the dynamics of $\mathcal{P}$.

The problem of determining when an embedding $\mathcal{P} \subset \hat{\mathcal{P}}$ will not disturb the dynamics of $\mathcal{P}$ is the problem of studying how the products $p_{ij}c^*_{ij}$ restrict, or effectively contract, the distribution of strength to proper subsets of $\hat{\mathcal{P}}$ under specific input paradigms to $\mathcal{P}$ at any chosen time. Under complicated input paradigms, the set $\mathcal{P}$ which we are naturally let to consider might itself change through time, so that the problem becomes one of studying embeddings $\mathcal{P}(t) \subset \hat{\mathcal{P}}$. For example, consider any set $\hat{\mathcal{P}} = \{p_1, p_2, \ldots, p_n\}$ of points such that every p_i is line-connected to at least one other point of $\hat{\mathcal{P}}$. Subject $\hat{\mathcal{P}}$ to a sequence of external inputs that runs cyclically through all of $\hat{\mathcal{P}}$ in the indexed order. In this situation, we cannot hope to isolate a fixed Q, which is never dynamically important. Rather, we must examine subsets $\mathcal{P}(t)$ which surround that point to which the last input was delivered. In fact, the closer the line structure approximates $p_{ij}c^*_{ij} = 0$ whenever $j \neq i+1(\text{mod } n)$, the greater the dynamical contraction will be in this $\hat{\mathcal{P}}$, which is, of course, the $\hat{\mathcal{P}}$ of serial verbal learning. In order to emphasize the importance of a given external input paradigm $\mathcal{J}$ on a field $\mathcal{F}(\hat{\mathcal{P}})$ with point set $\hat{\mathcal{P}}$, we shall often denote the field so determined by the pair $(\mathcal{F}(\hat{\mathcal{P}}), \mathcal{J})$.

Either as a result of structural isolation of a point by the proper choice of small p_{ij} values or by the approximation of the line structure to the configuration $c^*_{ij} \approx 0$ whenever $j \notin U_i$, where U_i is a small set of indices including $i+1(\text{mod } n)$, many of the stability properties of a small field can be expected even in a large field under a global cyclical input schema. We infer that dramatic restrictions of p_{ij} values hardly ever occur in simple verbal learning by the fact that all transitions between matched verbal units are possible to learn. The search for the most effective dynamical contractions in serial verbal learning is thus reduced to an examination of those cyclic input sequences under which the c^*_{ij} functions contract most rapidly to some analog of the asymptote $c^*_{ij} = \delta_{j,i+1}$, $i \leq n-1$.

8. A Rule for Responding.

We earlier postulated that an external input to a p_i with $c^*_{ij} \approx 1$ will generate the response r_j with high probability. The intuitive reason for this was that practically all of the transmitted strength at p_i will be funnelled to p_j over l_{ij}, whence $s_k << s_j$, $k \neq i$, and the various p_k, $k \neq i$, will exert only trivial dynamical effects on the field. We can extend this postulate to say that whenever an input generates strength transmission to a small set of points, responses from among these points will be more probable than responses from the points which have received only meager transmissions. The intuitive rule becomes: the most effective dynamical contractions generate the most stable responding. Such a rule is completely in keeping with our original reason for studying dynamical contractions. We saw that under an embedding of $\mathcal{P}$ into $\hat{\mathcal{P}}$ we could preserve the dynamics of $\mathcal{P}$ only by choosing the $p_{ij}c^*_{ij}$ functions restrictively. The present rule extends this observation to say that those points whose interactions are the most highly contracted will simultaneously be the points whose dynamics are most resistant to extensions of $\mathcal{P}$ to $\hat{\mathcal{P}}$ and the points most likely to generate responses upon being excited. The points whose excitation by an external input generates the most uniform strength distributions across the field are simultaneously the points which are the least likely to remain dynamically unchanged under extensions of $\mathcal{P}$ to $\hat{\mathcal{P}}$ and the points which have the least behavioral relevance to the subject. The field parsimoniously tends to preserve only those interactions which have arisen through experience. In the case of serial verbal learning, for example, realizing the asymptote $c^*_{ij} = \delta_{j,i+1}$, $j \leq n-1$, means both that the presentation of an input to p_i will always generate the correct response r_{i+1} and that the dynamics of $\mathcal{P} = \{p_1, p_2, \ldots, p_n\}$ will remain maximally stable under the addition to the field of new points and their interactions.

9. Towards a Resolution of Contiguity and Gestalt Theories: Local Strength Fields Do Not Suffice

Achieving a line asymptote like $c^*_{ij} = \delta_{j,i+1}$ is not sufficient to insure stable responding under all cyclic input sequences to p_1, then to p_2, and so on. The length w of the time interval between successive inputs must also be considered. As w decreases to small values, the number of points which simultaneously have high strength values increases, the dynamical contraction becomes less effective, and responding becomes less stable. The ordering of the inputs and all of the initial values of field quantities are the same for all values of w, yet the response distribution varies with w.

By preserving the ordering of inputs as w varies, we have preserved all "contiguity" relations between the points. By preserving all initial values, we have insured that we always begin with the same field, whatever the w. Any classical contiguity theory must predict that all choices of w generate the same response distribution. This does not occur either in the laboratory or in an embedding field. No classical contiguity theory can be adequate. This criticism is not restricted to contiguity theories, but extends to any psychological theory which does not display a running time variable in a prominent place.

It is nonetheless true that one can speak meaningfully in terms from the vocabulary of a contiguity theorist in discussing some of the simplest properties of the c_{ij} and c_{ij}^* functions. But contiguity considerations break down even if we merely alter the temporal relations between the inputs and leave the initial field invariant, or just alter the initial field and leave the input paradigm invariant. Such variations always allow us to find strength distributions which involve interactions over broad portions of the field. When this occurs, we can picturesquely say that the field behaves in a peculiarly Gestaltist way. Part of the tenacity of some of the most perplexing controversies between Gestalt field theorists and peripheralists in the past is doubtless due to the fact that the structures which they studied actually are capable of exhibiting effects which are sometimes compatible with the one theoretical perspective, sometimes with the other. The centrality of this issue and the impossibility of deciding it in favor of any one theoretical group are mirrored by the fact that even our simplest embedding field can exhibit both types of behavior under very natural variations in field conditions. Recognizing that the difficulty lies more in a poverty of theoretical conceptions than in the need for a real choice, let us turn to a finer study of the interplay of these two types of effects.

10. Concentration Sets and Virtual Points

Again imagine a serial nonsense syllable situation with points $p_1, p_2, \ldots, p_n$ labelled in the order of input presentation. Assume the usual symmetries in initial values, with $p_{ij} = 1 - \delta_{ij}$, and suppose that the time between successive inputs (the intratrial interval) is the same for all items of the list but (possibly) the last and the first item which usually enjoy a longer interval between onset times (the intertrial interval). Define the function $R_{ij} = \min(|i-j|, |n+i-j|, |n+j-i|)$ as a measure of the temporal remoteness of pairs of points relative to the cyclic input sequence, denote the point to which the most recent input has been delivered up to time t by $P = P(t)$, and let $R_{Pk} = R_k$. Suppose that the intratrial interval w is fixed relative to the theoretical constants of the embedding field at a presentation rate corresponding to, say, a two second empirical rate; that is, adjust w until it is larger than the largest intratrial intervals for which the c^*_{ij} structure shows only very small fluctuations under cyclic repetitions of the input sequence. Also suppose that the intertrial interval is at least $2w$ in length. We observed in briefly comparing contiguity and Gestaltist perspectives that varying w causes nontrivial effects on the field. We study this and related phenomena now, by setting w at steadily increasing values and considering the strength distribution over all points in the field during a complete single cyclic run through the list at each fixed value of w. A rigorous way to do this is to suppose that a copy $\mathcal{F}^w(\mathcal{P})$ of the field $\mathcal{F}(\mathcal{P})$ is given for every w. Present one cycle of the input sequence to $\mathcal{F}^w(\mathcal{P})$ with an intratrial interval of w. Let s_k^w be the resulting strength function of the p_k point in $\mathcal{F}^w(\mathcal{P})$. As w increases, an increasing proportion of $\Sigma_k s_k^w$ will be "concentrated about P", in a sense which we now explicate:

For any point p_i, let $\overline{p}_i = i$ and choose $\epsilon > 0$. Let $G_\epsilon(P, t)$ be the maximal set of the form $(p_{\overline{P}-i}, p_{\overline{P}-i+1}, \ldots, p_{\overline{P}}, p_{\overline{P}+1}, \ldots, p_{\overline{P}+j})$ of points p_k for which $s_k(t) \geq \epsilon s_{\overline{P}}(t)$, and let $|G_\epsilon(P, t)|$ be the number of points in $G_\epsilon(P, t)$. $G_\epsilon(P, t)$ is called the ϵ-<u>concentration set about</u> P <u>at time</u> t. As a function of P, $|G_\epsilon(\cdot, \cdot)|$ increases as P moves from the beginning to the middle of a single run of the input sequence for each w and all ϵ. This happens because the strength residues and the induced transmissions of the first few inputs accumulate as more inputs occur. After P moves from the middle towards the end of the input cycle, $|G_\epsilon|$ will begin to decrease. This decrease is due to the fact that by choosing the intertrial interval sufficiently long, no new inputs are delivered to the field while the strength values for points near the beginning, middle and, later, at the end of the list have a chance to decay. When considering G_ϵ for P near the end of the list, it is useful to imagine that after $P = p_n$, external inputs continue to be delivered, perhaps several times in succession, to an additional "virtual" point p^* at the w rate. p^* does not at all interact with the p_i and has no dynamical significance apart from the convenience of stating: it is while $P = p^*$ that the $|G_\epsilon|$ shrinkage at the end of the list occurs, and all remarks concerning special effects at the "end" of a single cyclic input sequence refer to times when $P = p^*$. This fact might be found slightly subtle because of its complete triviality: if P did not become trapped in p^* after $P = p_n$, then p_n could not be the end of the list, under our assumption that the intertrial interval is strictly greater than the intratrial interval. While if the intertrial interval is not strictly greater than the intratrial interval, the shrinking effect at the list's end, if it occurs at all, will occur for entirely different reasons, to which we shall presently come. The $|G_\epsilon|$ shrinking effect at the end of a single input cycle is simply due to the additional opportunity for strength decay near the list's beginning, middle, and finally its end, that a longer rest period between inputs at the list's end provides.

For increasingly large settings of w, $|G_\epsilon|$ as a function of P will approach a minimal constant which depends on the constants of the field equations and inputs, and on ϵ. It is in this sense that $\Sigma_k s_k^w$ concentrates about P as w increases. For intermediate values of w, the growth and decay of $|G_\epsilon|$ as a function of increasing P, for variable choices of ϵ, will often be accompanied by the general tendency for $c^*_{\overline{P}, \overline{P}+1}$ to decrease as P proceeds from the beginning to the middle of the list and then to increase as the end of the list is reached. For as increasing ϵ's are chosen, $G_\epsilon(P, t)$, with P and t fixed, shrinks to a set which continues to contain $p_{\overline{P}+1}$ even as other points are eliminated, but the rate with which other points are eliminated is usually smaller near the middle of the list than at its ends. The strength distribution for P near the ends of the list has less mass and is more peaked than for P just beyond the list's middle (diagram 6).

A concentration set of lines $L_\epsilon(P, t)$ is defined in a similar way relative to the set $\{c^*_{\bar{P}j}\}$ by $L_\epsilon(P, t) = \{l_{\bar{P}j}: c^*_{\bar{P}j}(t) \geq \epsilon\}$. If $|L_\epsilon(P, t)|$ denotes the number of its elements, then as a function of increasing P, for small ϵ, $|L_\epsilon|$ will first increase and then decrease whenever the input paradigm is a single run of a cyclic input sequence with relatively long intertrial interval and intermediate values of w. The dilation, then contraction, of $|L_\epsilon|$ with increasing P is the exact counterpart of the decrease, then increase, of $c^*_{\bar{P}, \bar{P}+1}$ with increasing P. This behavior of $|L_\epsilon|$ for intermediate values of w is an example of an effective dynamical contraction in the normalized line structure near the ends of the list relative to the list's middle. We will see that this contraction in the lines in turn generates a similar contraction in the strength distribution about P induced by repeated input cycles. The relative contraction of the strength field produces many of the known bowing phenomena in serial learning.

Before we proceed to establish this connection, notice that all discussions of the "middle" of the list must be understood relative to the size of w and the size n of the field. For sufficiently large w and sufficiently small n, with no change in the growth and decay constants of the field, the dilation of $|L_\epsilon|$ in the list's "middle" is minimal, and "one-trial learning" of the list occurs. For the ranges of w and n customarily used used in serial experiments, however, w is so small and n so large that significant relative contractions are in fact produced.

11. Bowing Phenomena in Serial Learning

After the fairly long intertrial interval, the nearly exponentially decaying point strength values arising from the first input cycle will have subsided so completely that the line structuring induced by a second input cycle presentation will be relatively unconfounded by strength residues from the first presentation. The general form of the concentration functions will therefore be preserved on the second trial. The various relative dilations and contractions in the concentration functions might well be accented even further on the second trial by the nonuniform residue of line structure remaining from the first input cycle. For when an input is delivered during the second cycle, the strength will be transmitted through a more highly concentrated normalized line structure than on the first cycle to a subset of points whose point strength values will grow to considerably higher maxima than those of the points not in the concentration sets. As a result, the line functions in the concentrated set will grow correspondingly faster and will usually induce a parallel growth in the normalized lines. By continuing this argument over successive repetitions of the cyclic input sequence, we see that a progressive effective dynamical contraction in the lines occurs until $c^*_{i,i+1}$ far exceeds $c^*_{i,j}$, $j \neq i+1$, for all $i \neq n$. Nonetheless, letting $\Delta(i,\epsilon)$ be the first time at which $c^*_{i,i+1} \geq 1-\epsilon$, $0 \leq \epsilon \leq 1$, and letting $|\Delta(i,\epsilon)|$ be the number of inputs delivered to p_i at times $t \leq \Delta(i,\epsilon)$, $|\Delta(i,\epsilon)|$ as a function of i will often take its maximum near the middle of the list and its minimum near the beginning of the list, with intermediate values for the end. In terms of the number of trials necessary to reach a criterion of $1-\epsilon$ in the normalized lines, we are inclined to say that the middle of the list will be the hardest part to learn.

12. Remote Associations and Asymptotic Chaining in Serial Learning

The gradual contraction of the line concentration set to the linear asymptote $c^*_{i,i+1} = 1$, for $i \neq n$, helps to explain standard facts about the distribution of remote associations over a given trial and for increasing numbers of trials. The "strength of associations" from p_i to p_j is again measured in this elementary discussion by c^*_{ij}. The remoteness of p_j from p_i is roughly given by R_{ij}. R_{ij} is a convenient but crude measure of remoteness since it accounts only for ordering relations among the p_i's and omits reference to the difference between intertrial and intratrial intervals.

To offset this difficulty, include p^* in the list as $\{p^*_{n+1}, p^*_{n+2}, \ldots, p^*_{n+k}\}$, where k is the greatest number of integral multiples of the intratrial interval into which the intertrial interval can be decomposed. Measure R_{ij} with respect to the augmented list $\{p_1, p_2, \ldots, p_n, p^*_{n+1}, p^*_{n+2}, \ldots, p^*_{n+k}\}$ in order to include temporal effects. No lines join the p_i with the various virtual points p^*_{n+j}, which we include merely to conveniently compare effects produced at some p_m on one trial with effects produced at another p_m on some other trial.

Fix attention on some p_i. As the linear asymptote is progressively reached with successive input cycle presentations, the bulk of the strength transmitted from p_i to points p_k for which R_{ik} is large will increasingly have to be transmitted to p_k in several steps by way of less remote points. For on any one trial, the c^*_{ik} values generally decrease with increasing R_{ik} in an initially homogeneous field. With the passage of successive trials, the c^*_{ik} for which R_{ik} is largest decrease the fastest, the c^*_{ik} with R_{ik} in an intermediate range decrease less quickly, and so on. As the linear asymptote is approached and the more remote points become only weakly associated with p_i, the strength transmissions flow very much as they did when we chose $p_{ij} = \delta_{j, i+1}$, for $j \leq n-1$. Asymptotically, a chain of successive transmissions becomes necessary to excite a remote point. We have already noted that k-step strength transmission in such a chain is killed off at a rate faster than r^k. As learning proceeds, the frequency of remote associations will thus approach a negatively accelerated function of R_{ik}. After many trials, only a few p_j near to p_i will receive nontrivial strength transmissions when an input is delivered to p_i. The set of remote associations gradually shrinks to sets like $\{p_{i-1}, p_{i+1}, p_{i+2}\}$, and finally to $\{p_{i+1}\}$ alone, at the same time that the negatively accelerated distribution of remote associations is approached. The exponential local decay of point strength at a uniform rate β across points will also contribute to the successive approximation through time of a negatively accelerated strength distribution process to increasingly small sets of points.

Implicit in these remarks is the fact that a similar process holds throughout for the forward and the backward line structures. Usually $c^*_{i,i+1} > c^*_{i,i-1}$, $i \neq 1, n$, by an obvious extension of the argument used to show that

$c_{12}^* > c_{21}$ in the four point field. The backward effects will usually be smaller than the forward effects, but similarly distributed. In particular, for a p_i chosen near the middle of a sufficiently long list, remote associations in the forward direction will often include more points and be more frequent than remote associations in the backward direction. A convenient way to discuss such effects is to let the set of points $\{p_1, p_2, \ldots, p_{\bar{P}(t)}\}$ be called the past field of the input sequence on a given trial, and to call $\{p_{\bar{P}(t)+1}, p_{\bar{P}(t)+2}, \ldots, p_n\}$ the future field. When $P(t)$ is near the middle of the list, we can usually pair off lines to the future field with lines to the past field so that the line strength of the future field line exceeds that of the past field line.

Our discussion up to now has been concerned with simple thought experiments that illustrate important features of the embedding equations which are compatible with experience. These examples do not nearly offer a complete protocol of the numerous effects that can arise under an actual input sequence, but as more of them are considered, the reader's intuition into the sources of these effects and their interrelationships should correspondingly increase.

13. Feedback and Random Inputs and the Sharing of Stimulus and Response Properties

Some slightly more subtle dynamical effects will now be considered. We require that whenever the subject utters a consonant r_i, whether as a guess of a forthcoming list item or merely to say the stimulus item aloud, an input is delivered to p_i. The onset time of this input will be shifted slightly in time from the onset of the evoked response, but we will ignore this shift in our qualitative discussion. Those inputs which represent the saying of a consonant immediately after it is presented by the experimenter shall again be called experimental inputs. All other inputs are called feedback inputs. We assume for simplicity that the functional form of the two types of inputs corresponding to a subject's utterances is the same, and put the discussion of this assumption aside until we are prepared for it. For the present, simply let $J_i^{(e)}$ be determined as before, and write $a_i^{(e)}$ and $b_i^{(e)}$ to distinguish these constants of $J_i^{(e)}$ from $a_i^{(f)}$ and $b_i^{(f)}$, which occur in $J_i^{(f)}$, the feedback input. (f) will always be used to mark a feedback quantity. The total input to p_i now takes the form

$$\Sigma_j J_i^{(e)} (t - s_{ij}^{(e)}) + \Sigma_k J_i^{(f)} (t - s_{ik}^{(f)}),$$

where $s_{ik}^{(f)}$ is the time at which the k^{th} feedback input is delivered to p_i. In an initially homogeneous field under a serial input paradigm, we can let $a_i^{(e)} = a_j^{(e)}$, $a_i^{(f)} = a_j^{(f)}$, and so on.

This assumption about feedback inputs implies that responses enjoy many stimulus properties. In the serial verbal learning situation, this is particularly true. It has sometimes been argued to the contrary that guessing responses, in particular incorrect guesses, should not be viewed as self-induced stimuli since they do not always seem to influence the response record. We shall show that the fact that feedback inputs are delivered does not always entail a noticeable change in the response record, and shall delineate those situations for which this is true. The argument that feedback inputs do not exist because the response record does not always change will hereby be shown to be a nonsequitor. In fact, this particular nonsequitor is closely bound to an enduring controversy between "all-or-none" theorists and "continuity" theorists of learning which is fed by a misunderstanding of the way in which inputs interact with field structure in general. The combatants in this controversy, as in the Contiguity vs. Gestaltist controversy, have armed themselves with an incomplete set of theoretical variables. We shall return to this matter after making some preliminary observations based on the existence of feedback inputs.

Since we have assumed that a subject's guesses directly perturb the field, it is important to state more clearly the rules determining when a subject will guess. This we shall do very roughly now to permit a discussion which will lead naturally to a more precise determination of these rules later on. Suppose that c^*_{ij} increases on successive trials. After successive experimental inputs to p_i, s_j will usually grow to ever higher asymptotes. If s_j exceeds a certain prescribed size Υ at a time when it majorizes all other s_k, $k \neq i$, we assume that the subject will utter the consonant r_j. This utterance corresponds to the delivery of a feedback input to the point p_j. Assume for simplicity that no more than one response can occur between any pair of successive experimental inputs.

We now add another member to the family of field inputs: a random input $J_i^{(r)}$ which represents a small fluctuating input such that $\Sigma_i J_i^{(r)}$ is equally distributed in the mean among all p_i in a homogeneous field. $J_i^{(r)}$ is a type of internal input caused by events in regions of the subject's brain which are not represented by $\mathcal{F}\{p_1, \ldots, p_n\}$ and which must consequently be considered as random fluctuations until we are ready to extend $\mathcal{F}\{p_1, \ldots, p_n\}$. The $J_i^{(r)}$ are introduced at this time as a natural complement of the functions $J_i^{(f)}$, which represent the "external" perturbations which $\mathcal{F}\{p_1, \ldots, p_n\}$ sends to itself.

14. Error Distributions

Using these extended postulates, we can interpret our previous remarks on c^*_{ij} functions in terms of response distributions. As usual, once $c^*_{i,i+1} \approx 1$ for all $i \neq n$, the presentation of an isolated input to a p_j, $j \neq n$, will induce one-step transmission almost exclusively to p_{j+1}, and this transmission process will tend both to preserve the normalized line values and to lead to the evokation of the correct response r_{j+1}. For a point p_k near the middle of a long list, the remarks on remote associations become: Anticipatory errors after an external input to p_k will be more frequent than backward errors as a result of the forward bias in the c^*_{kj} structure. Before the c^*_{kj} functions contract from their originally homogeneous state, however, errors might well be scattered broadly about the list. This does not mean that large quantities of errors will surely occur, but only that the distribution of errors will tend to be quite uniform if errors do occur. As the line concentration functions contract through time, the distribution of errors will contract as well, and will often approximate a distribution that decreases almost multiplicatively with remoteness due to the successive approximation of the normalized line structure to a cyclic chain. One effect of this contraction is the enhancement of the $c^*_{\overline{P}j}$ values for small R_j, since the maxima of these s_j, given an input to P, will be larger than the maxima attained within a field of the same size with uniform line structure. The persistent evokation of a nearby anticipatory error is the frequent behavioral correlate of this phenomenon; for example, r_{i+2} is given instead of r_{i+1} in response to an experimental input to p_i.

These remarks have a counterpart for positions near the beginning and the end of the list. For example, the backward line structure is often relatively stronger than the forward line structure for a point p_i at the end of a list than for a point p_k at the list's middle. The response distribution will pattern itself in close correspondence with these relative differences in distribution. Such inequalities as $c_{i,i-1}/c_{i,i+1} > c_{k,k-1}/c_{k,k+1}$ do depend delicately on the ratio of the intertrial and intratrial intervals, but a quantitative study of this dependence is beyond the scope of this introductory discussion.

15. Mixing Distributed and Massed Practice

Distributed practice will be relatively more advantageous than massed practice at the beginning of learning than at its end. For we have already seen that, given a homogeneous field, increasing w causes a decrease in $|L_\epsilon|$ as a function of P. Distributing practice (choosing w large) at the beginning of learning therefore contracts the instantaneous strength distribution generated about P by external inputs, whence the linear line asymptote is more rapidly approximated, and more rapid learning is achieved. Once the normalized line structure is partially contracted, it can better channel the incoming experimental inputs than can a homogeneous field, and thereby reduces the homogenizing effect that widespread simultaneous strength activity has on the normalized line structure.